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Some stability properties of linear operator equations

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I. INTRODUCTION

In this paper we consider a closed linear operator A with domain D(A) contained in a Banach space X and range R(A) contained in a Banach space Y. We assume that R(A) is closed. Our primary investigation is closely related to the problem of determining whether or not R(A + C) is closed for some linear operator C. We shall use $\alpha(A)$ to denote the dimension of the null space of A, N(A), and $\beta(A)$ to denote the dimension of the quotient space Y/R(A). It is known that if either $\alpha(A)$ or $\beta(A)$ is finite then R(A + C) is closed whenever the norm of the operator, $|C|$, is sufficiently small (Kato, 6, p. 281). We are interested in extending these results to the case where $\alpha(A)$ and $\beta(A)$ are both infinite. However, it follows from a result of Goldman (4) that this is not possible in general. We shall investigate conditions under which $R(A - \lambda B)$ is closed for all λ in a neighborhood of 0 where B is some linear operator.

In Chapter II we define a sequence of linear manifolds $M_n(A:B)$ and a linear manifold J(A:B) which is the intersection of the manifolds $M_n(A:B)$ for all n. Kato (6, p. 297) has shown that if B is A-bounded (i.e. $|Bx| \leq k(|x| + |Ax|)$ for all $x \in D(A)$ where k is a positive constant) and if $N(A) \subset J(A:B)$ then $R(A - \lambda B)$ is closed for all λ in a neighborhood of 0. We state this result in Theorem 2.8. Throughout the paper we shall assume that B is A-bounded. In Theorem 2.9 we assume that $B(N(A)) = \{Bw : w \in N(A)\}$ is closed and that $R(A)$ and $B(N(A))$ are "strongly disjoint" (i.e. $|y + Bw| \geq k|y|$ for all $y \in R(A)$ and all $w \in N(A)$ where k is a positive constant). Under these conditions again $R(A - \lambda B)$ is shown to be closed for all λ in some neighborhood of 0. We conclude the chapter with an example which shows that the conclusion of Theorem 2.9 does not necessarily follow if the strongly disjoint condition is weakened to a "disjoint" condition (i.e. $R(A) \cap B(N(A)) = \{0\}$).

In Chapter III we define a linear manifold $K(A:B)$ which we show in Theorem 3.1 to be equivalent to $B(J(A:B))$. The main object of this chapter is to show that for all λ in certain regions of the complex plane, $K(A - \lambda B:B)$ is independent of λ. In Theorem 3.7 we show that this is the case for a connected open set of complex numbers, Γ, which has the property that $N(A - \lambda B) \subset J(A - \lambda B:B)$ for all $\lambda \in \Gamma$. Furthermore by Theorem 3.8 for every $y \in Y$, either $y \in K(A - \lambda B:B) \subset R(A - \lambda B)$ for all $\lambda \in \Gamma$, or the set of all $\lambda \in \Gamma$ for which $y \in R(A - \lambda B)$ has no limit point in Γ. We continue in Theorem 3.9 to demonstrate that for $\lambda \in \Gamma$, $\alpha(A - \lambda B)$ and $\beta(A - \lambda B)$ are constant. Theorem 3.15 shows that for a connected open set of complex numbers, Γ_1, which has the property that either $\alpha(A - \lambda B)$ or $\beta(A - \lambda B)$ is finite for all $\lambda \in \Gamma_1$, again

$K(A - \lambda B:B)$ is independent of λ in Γ_1. We conclude Chapter III with Theorem 3.20 in which we show that if $B(N(A))$ is closed and if $R(A)$ and $B(N(A))$ are strongly disjoint, then $K(A - \lambda B:B)$ is independent of λ in a deleted neighborhood of 0. Furthermore we show that $K(A - \lambda B:B)$ is the direct sum of $K(A:B)$ and $B(N(A))$ for all λ in this neighborhood.

In Chapter IV we show that the linear manifolds, $AM_n(A:B)$ are closed for all n under any one of the following conditions: (1) $N(A) \subset J(A:B)$, (2) $\alpha(A)$ is finite, (3) $\beta(A)$ is finite, (4) $B(N(A))$ is closed and strongly disjoint from $R(A)$. It follows that $K(A:B)$ is also closed under any of the four above mentioned conditions.

Chapter V is devoted to demonstrating that if $N(A)$ and $J(A:B)$ are strongly disjoint, then the inverse of $A - \lambda B$ exists for all λ in some deleted neighborhood of 0.

We devote Chapter VI to the special case where $X = Y$ and $B = I$ (the identity operator). In this case $K(A:B)$ reduces to the "Kernmannigfaltigkeit" first introduced by F. Riesz (8, p. 87) and which has been studied by Gokhberg and Markus (3), Gokhberg and Krein (2) and Homer (5).

We consider the concepts of regular extension which have been studied by Homer (5). Theorem 6.3 along with a result of Homer give a necessary and sufficient condition for a regular extension at λ to be a regular extension near λ.

We conclude by discussing the relationships between $N(A)$,

$R(A)$ and $K(A:I)$ which are sufficient to insure that $R(A - \lambda I)$ is closed for all λ in some neighborhood of 0. Applying the results of Chapter I we have that $R(A - \lambda I)$ is closed for all λ near 0 if either $N(A) \subset K(A:I)$ or $N(A)$ is strongly disjoint from $R(A)$. The question of what happens when $N(A) \subset R(A)$ but $N(A) \cap K(A:I) = \{0\}$ is considered at this point. By example we show that this case may result in $R(A - \lambda I)$ being not closed for every $\lambda \neq 0$.

Very briefly, we would like to look at the possibilities of further generalizing the results of this paper. (For simplicity we continue to use the special case considered in Chapter VI.) Suppose that $N(A)$ can be decomposed into three linear manifolds L_1, L_2 and L_3 such that $N(A) \doteq L_1 \oplus L_2 \oplus L_3$ where $L_1 \subseteq K(A:I)$ and L_2 is strongly disjoint from $R(A)$. It seems likely that if L_3 is finite dimensional the main results of this paper could be obtained. The possibilities along this line have not yet been investigated. However, our examples demonstrate that in the absence of this type of structure not much can be expected.

II. OPERATORS WITH CLOSED RANGE

Throughout this paper we shall consider two complex Banach spaces X and Y and a pair of linear operators A and B. We assume that $D(A) \subset D(B) \subset X$ and that the range of both operators is contained in Y. We shall assume that for the complex number λ_0, $(A - \lambda_0 B)$ is a closed linear operator with closed range and that $R(A - \lambda_0 B) \neq \{0\}$. We further assume that there exist non-negative numbers σ and τ such that for all $x \in D(A)$ the following inequality is satisfied:

$$|Bx| \leq \sigma|x| + \tau|(A - \lambda_0 B)x|.$$

We shall refer to the conditions stated above as the general hypothesis.

In this chapter we concern ourselves primarily with the existence of a neighborhood of $\lambda_0 = 0$ such that $R(A - \lambda B)$ is a closed linear manifold for all λ in the neighborhood. Our general hypothesis is not sufficient to guarantee the existence of such a neighborhood as we shall show by example. We begin our investigation by reviewing certain finiteness conditions and a restriction on $N(A)$ which Kato (6) has shown to be sufficient for the existence of such a neighborhood. A new theorem of similar nature is then presented and we conclude the chapter with the above mentioned example.

We shall need to consider several quotient spaces in this paper. If L is any linear manifold contained in X we use x' to denote the equivalence class in X/L which contains the element x. That is, $x' = \{x + t : t \in L\}$. If L is a closed linear manifold then $|x'| = \inf\{|x + t| : t \in L\}$ can be shown to be a norm on the quotient space X/L and the space with this norm is a Banach space (Taylor, 9, p. 105). Whenever L is closed we shall consider X/L as a Banach space and assume that the norm is the one mentioned above.

Definition 2.0: For a linear operator T with $R(T) \subset Y$ we define $\beta(T)$ to be the dimension of the quotient space $Y/R(T)$.

Definition 2.1: For a linear operator T we define $\alpha(T)$ to be the dimension of $N(T)$.

In considering the dimension of a linear space or subspace we shall not distinguish between the various infinite cardinal numbers.

Kato (6, p. 316) has shown that if either $\alpha(A)$ or $\beta(A)$ is finite then there exists a neighborhood of 0 such that $R(A - \lambda B)$ is closed for all λ in this neighborhood.

Before proceeding to the case which allows both $\alpha(A)$ and $\beta(A)$ to be infinite we shall need to consider a number of preliminary definitions and lemmas.

If T is any linear operator and S is any set it will be convenient to use TS to denote the set of all Tx where $x \in S \cap D(T)$. We use $T^{-1}(S)$ to represent the set of all x for which $Tx \in S$. We do not imply that the inverse operator T^{-1} exists.

<u>Definition 2.2</u>: For two linear operators T and U such that $D(T) \subset D(U) \subset X$ and the range of both operators is contained in Y, we define a sequence of linear manifolds, $M_n = M_n(T:U)$, as follows:

$$M_0 = X$$
$$M_n = U^{-1}(TM_{n-1}) \qquad n = 1, 2, 3, \cdots$$

<u>Definition 2.3</u>: For such T and U we define the linear manifold $J(T:U)$ to be the intersection of the linear manifolds, $M_n(T:U)$, for all non-negative n.

We remark that since $U^{-1}\{0\} = N(U)$ and since $0 \in TM_n$ for all n, $N(U) \subset M_n$ for all n. Thus $N(U) \subset J(T:U)$.

<u>Definition 2.4</u>: For any linear operator T with $D(T) \subset X$ and $R(T) \neq \{0\}$ we define the operator T' on $\frac{D(T)}{N(T)} \subset \frac{X}{N(T)}$ by the equation $T'x' = Tx$.

We remark that T' is linear, one-to-one and $R(T') = R(T)$. The following lemma is well known.

<u>Lemma 2.5</u>: If T is closed then T' is closed.

Proof: If T is closed, $N(T)$ is closed so $X/N(T)$ is a Banach space. Let $x_n' \rightarrow x'$ and $T'x_n' \rightarrow y$. From the definition of the norm in $X/N(T)$ there exists a sequence $\{z_n\} \subset N(T)$ such that $x_n - x + z_n \to 0$. $T(x_n + z_n) = Tx_n = T'x_n' \to y$ and $x_n + z_n \to x$ imply that $x \in D(T)$ and $Tx = y$ since T is closed. Therefore $x' \in D(T')$ and $T'x' = y$.

Definition 2.6: We define the quantity $\gamma(T)$ as follows:

$$\gamma(T) = \frac{1}{|(T')^{-1}|}$$

if $(T')^{-1}$ is bounded and $\gamma(T) = 0$ otherwise.

We remark that $|(T')^{-1}| \neq 0$ since $R(T) \neq \{0\}$ so $\gamma(T)$ is well defined.

The following lemma is also known.

Lemma 2.7: If T is closed then $\gamma(T) > 0$ if and only if $R(T)$ is closed. Furthermore $|T'x'| \geq \gamma(T)|x'|$ for all $x' \in D(T')$.

Proof: By Lemma 2.5, T' is closed so $(T')^{-1}$ is closed. From Definition 2.6, $\gamma(T) > 0$ is equivalent to $(T')^{-1}$ bounded. But $(T')^{-1}$ bounded is equivalent to $D[(T')^{-1}]$ closed since $(T')^{-1}$ is closed. Now $D[(T')^{-1}] = R(T') = R(T)$ so we see that $\gamma(T) > 0$ is equivalent to $R(T)$ is closed. Now let $x' \in D(T')$ and let $y = T'x'$. Then $x' = (T')^{-1}y$. If $\gamma(T) > 0$, $|(T')^{-1}y| \leq |(T')^{-1}||y|$ which implies that $|y| \geq \gamma(T)|(T')^{-1}y|$. Thus $|T'x'| \geq \gamma(T)|x'|$. If $\gamma(T) = 0$ the inequality is clearly true.

We are now ready to consider another result of Kato (6) the proof of which is beyond the scope of this paper.

<u>Theorem 2.8</u> (6, p. 297): If $N(A) \subset J(A:B)$[1] and λ satisfies the inequality:

$$|\lambda| < \frac{\gamma(A)}{3\sigma + \tau\ \gamma(A)}$$

then $A - \lambda B$ is a closed linear operator with closed range. Furthermore $\alpha(A - \lambda B)$, $\beta(A - \lambda B)$ are constant,

$$N(A - \lambda B) \subset J(A - \lambda B : B)$$

and $\gamma(A - \lambda B) \geq \gamma(A) - (3\sigma + \tau\ \gamma(A))|\lambda|$.

We note that $N(A) \subset J(A:B)$ along with our general hypothesis is sufficient to insure that $R(A - \lambda B)$ is closed for all λ in a neighborhood of 0 regardless of whether or not $\alpha(A)$ or $\beta(A)$ are finite.

We now state and prove a theorem which we believe to be new.

<u>Theorem 2.9</u>: Let $B(N(A))$ be closed and suppose there exists a positive number k such that $|Ax + Bw| \geq k|Ax|$ for all $x \in D(A)$ and all $w \in N(A)$. Then for all λ which satisfy the inequality:

$$0 < |\lambda| < \frac{k\ \gamma(A)}{\sigma + \tau\ \gamma(A)},$$

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[1]The condition $\nu(A:B) = \infty$ used by Kato is clearly equivalent to $N(A) \subset J(A:B)$.

$A - \lambda B$ is a closed linear operator with closed range. Furthermore $N(A - \lambda B) = N(A) \cap N(B)$ and $N(A - \lambda B) \subset J(A - \lambda B:B)$.

We find it convenient to state and prove a number of lemmas which will be used in the proof of this theorem.

Lemma 2.10 (Lorch, 7, p. 220): Let L_1 and L_2 be closed linear manifolds of a Banach space and let L be the set of all y such that $y = x_1 + x_2$ for some $x_1 \epsilon L_1$ and $x_2 \epsilon L_2$. If there exists a positive number k such that $|x_1 + x_2| \geq k|x_1|$ for all $x_1 \epsilon L_1$ and all $x_2 \epsilon L_2$ then $L_1 \cap L_2 = \{0\}$ and $L_1 \oplus L_2$ is a closed linear manifold.

We remark that although Lemma 2.10 was originally proved in a paper on reflexive vector spaces, the proof does not require reflexivity of the space. We also note that the positive number k will always be less than or equal to one $(k \leq 1)$.

We now consider two quotient spaces. Let $X' = X/N(A)$ and let $Y' = Y/B(N(A))$. Let θ be the canonical mapping of X onto X' and let ϕ be the canonical mapping of Y onto Y'. Both X' and Y' are Banach spaces since $N(A)$ and $B(N(A))$ are closed.

Definition 2.11: We define the operator $\hat{A}$ from $D(\hat{A}) = \frac{D(A)}{N(A)} \subset X'$ into Y' by the equation: $\hat{A}x' = \phi(Ax)$ where $x \epsilon x'$.

It is clear that $\hat{A}$ is a linear operator.

<u>Definition 2.12</u>: We define the operator $\hat{B}$ from $D(\hat{B}) = \frac{D(B)}{N(A)} \subset X'$ into Y' by the equation: $\hat{B}x' = \phi(Bx)$ where $x \in x'$.

To see that $\hat{B}$ is actually a function let $\theta(x_1) = \theta(x_2) = x'$. Then $x_2 - x_1 \in N(A)$ so that $Bx_2 - Bx_1 \in B(N(A))$ which implies that $\phi(Bx_2) = \phi(Bx_1)$.

It is clear that $\hat{B}$ is linear and we remark that $D(\hat{A}) \subset D(\hat{B})$ since $D(A) \subset D(B)$.

The following lemma is known.

<u>Lemma 2.13</u>: Let F be a linear manifold contained in the Banach space X and let L be a closed linear manifold such that $L \subset F$. Let μ be the canonical mapping of X onto X/L. Then $\mu(F)$ is closed if and only if F is closed.

Proof: Assume F is closed. Let $x' \in \overline{\mu(F)}$. Then there exists a sequence $\{x_n'\} \subset \mu(F)$ such that $x_n' \to x'$. From the definition of the norm in X/L there exists a sequence $\{z_n\} \subset L$ such that $x_n - x + z_n \to 0$ where $x \in x'$ and $x_n \in x_n'$ for each n. We can assume that $\{x_n\} \subset F$ and since $L \subset F$, $\{x_n + z_n\} \subset F$. Since $x_n + z_n \to x$, $x \in F$ because F is closed. Thus $x' \in \mu(F)$.

Now assume $\mu(F)$ is closed and let $x \in \overline{F}$. Then there exists $\{x_n\} \subset F$ such that $x_n \to x$ so that $\mu(x_n) \to \mu(x)$ and $\{\mu(x_n)\} \subset \mu(F)$. Since $\mu(F)$ is closed, $\mu(x) \in \mu(F)$. Therefore there exists $y \in F$ such that $\mu(x) = \mu(y)$ which implies that $x - y \in L$. But $L \subset F$ so $x \in F$.

<u>Lemma 2.14</u>: $R(\hat{A})$ is closed in Y'.

Proof: By Lemma 2.10, $R(A) \oplus B(N(A))$ is closed and by Lemma 2.13, $\phi(R(A) \oplus B(N(A)))$ is closed. It is clear that $\phi(R(A) \oplus B(N(A))) = \phi(R(A)) = R(\hat{A})$.

<u>Lemma 2.15</u>: (a) $\hat{A}$ is closed, (b) $\hat{A}$ is one-to-one, (c) $|\hat{A}x'| \geq k\gamma(A)|x'|$ and (d) $|Ax| \leq \frac{1}{k}|\hat{A}x'|$ for all $x' \in D(\hat{A})$.

Proof: Let x' be an arbitrary element of $D(\hat{A})$. Then $|\hat{A}x'| = |\phi(Ax)| = \inf\{|Ax + Bw| : w \in N(A)\}$. But

$$|Ax + Bw| \geq k|Ax|$$

for all $x \in D(A)$ and all $w \in N(A)$. Thus $|\hat{A}x'| \geq k|Ax|$ which proves (d) since x' is an arbitrary element of $D(\hat{A})$.

From Lemma 2.7, $|Ax| = |A'x'| \geq \gamma(A)|x'|$. This inequality combined with (d) gives (c). From this inequality we see that $\hat{A}$ is one-to-one (which proves (b)) and $(\hat{A})^{-1}$ is continuous. Since $D[(\hat{A})^{-1}] = R(\hat{A})$ is closed by Lemma 2.14, $\hat{A}$ is closed so (a) is true.

<u>Lemma 2.16</u>: If $|Bx| \leq \sigma|x| + \tau|Ax|$ for all $x \in D(A)$ then $|\hat{B}x'| \leq \sigma|x'| + \frac{\tau}{k}|\hat{A}x'|$ for all $x' \in D(\hat{A})$.

Proof: Let x' be an arbitrary element of $D(\hat{A})$. Then $|\hat{B}x'| = |\phi(Bx)| \leq |Bx| \leq \sigma|x| + \tau|Ax|$ holds for every $x \in x'$. From Lemma 2.15 (d), $|Ax| \leq \frac{1}{k}|\hat{A}x'|$ so $|\hat{B}x'| \leq \sigma|x| + \frac{\tau}{k}|\hat{A}x'|$ is true for every $x \in x'$. Thus

$$|\hat{B}x'| \leq \inf_{x \in x'} \{\sigma|x| + \tfrac{\tau}{k}|\hat{A}x'|\} = \sigma|x'| + \tfrac{\tau}{k}|\hat{A}x'|.$$

<u>Lemma 2.17</u>: If A is closed and there exist non-negative numbers σ and τ such that for all $x \in D(A)$,

$$|Bx| \leq \sigma|x| + \tau|Ax|,$$

then $A - \lambda B$ is a closed operator for all λ which satisfy the inequality: $\tau|\lambda| < 1$.

Proof: Let λ be any fixed complex number such that $\tau|\lambda| < 1$. Let $x_n \to x$ and $(A - \lambda B)x_n \to y$. Then

$$\begin{aligned}|(A - \lambda B)x_n - (A - \lambda B)x_m| &\geq |Ax_n - Ax_m| - |\lambda||Bx_n - Bx_m| \\ &\geq |Ax_n - Ax_m| \\ &\quad - |\lambda|(\sigma|x_n - x_m| + \tau|Ax_n - Ax_m|).\end{aligned}$$

Thus

$$|Ax_n - Ax_m|(1 - |\lambda|\tau) \leq |\lambda|\sigma|x_n - x_m| + |(A - \lambda B)x_n - (A-\lambda B)x_m|$$

and from this inequality it is clear that $\{Ax_n\}$ is a Cauchy sequence. Let

$$z = \lim_{n\to\infty} Ax_n.$$

Then $x_n \to x$ and $Ax_n \to z$ imply that $x \in D(A)$ and $z = Ax$ since A is closed. $|Bx_n - Bx| \leq \sigma|x_n - x| + \tau|Ax_n - Ax|$ from which we see that $Bx_n \to Bx$. Therefore $(A - \lambda B)x_n \to Ax - \lambda Bx = y$.

Lemma 2.18: $R(\hat{A} - \lambda\hat{B}) = \phi(R(A - \lambda B))$.

Proof: For each $x \in D(A)$ or equivalently for each $x' \in D(\hat{A})$ we see that

$$\phi((A - \lambda B)x) = \phi(Ax - \lambda Bx) = \phi(Ax) - \lambda\, \phi(Bx)$$
$$= \hat{A}x' - \lambda\hat{B}x' = (\hat{A} - \lambda\hat{B})x'.$$

From this equation the lemma is obvious.

Lemma 2.19: $B(N(A)) \subset R(A - \lambda B)$ for $\lambda \neq 0$.

Proof: Let $y \in B(N(A))$. Then there exists $w \in N(A)$ such that $y = Bw$. Thus $- w/\lambda \in N(A)$ for $\lambda \neq 0$. $(A - \lambda B)(- w/\lambda) = Bw = y$ so $y \in R(A - \lambda B)$.

We are now ready to complete the proof of Theorem 2.9. We note that

$$\frac{k\ \gamma(A)}{\sigma + \tau\ \gamma(A)} \leq \frac{k}{\tau} \leq \frac{1}{\tau}$$

since $k \leq 1$. Thus if

$$0 < |\lambda| < \frac{k\ \gamma(A)}{\sigma + \tau\ \gamma(A)},$$

$A - \lambda B$ is closed by Lemma 2.17.

We also note that Lemma 2.17 can be applied to $\hat{A} - \lambda\hat{B}$ since $\hat{A}$ is closed by Lemma 2.15 and $|\hat{B}x'| \leq \sigma|x'| + \frac{\tau}{k}|\hat{A}x'|$ for all $x' \in D(\hat{A})$ by Lemma 2.16. Thus $\hat{A} - \lambda\hat{B}$ is closed. Now

$$\begin{aligned} |(\hat{A} - \lambda\hat{B})x'| &\geq |\hat{A}x'| - |\lambda||\hat{B}x'| \\ &\geq |\hat{A}x'| - |\lambda|(\sigma|x'| + \tfrac{\tau}{k}|\hat{A}x'|) \\ &= |\hat{A}x'|(1 - |\lambda|\tfrac{\tau}{k}) - |\lambda|\sigma|x'| \\ &\geq k\ \Upsilon(A)|x'|(1 - |\lambda|\tfrac{\tau}{k}) - |\lambda|\sigma|x'| \end{aligned}$$

using Lemma 2.15 (c)

$$= |x'|(k\ \Upsilon(A) - |\lambda|(\tau\ \Upsilon(A) + \sigma)).$$

Thus for $0 < |\lambda| < \frac{k\ \Upsilon(A)}{\sigma + \tau\ \Upsilon(A)}$, $p = k\ \Upsilon(A) - |\lambda|(\sigma + \tau\Upsilon(A)) > 0$ and $|(\hat{A} - \lambda\hat{B})x'| \geq p|x'|$. Therefore $(\hat{A} - \lambda\hat{B})^{-1}$ exists and is continuous. Since $(\hat{A} - \lambda\hat{B})^{-1}$ is closed, $D[(\hat{A} - \lambda\hat{B})^{-1}] = R(\hat{A} - \lambda\hat{B})$ is closed. By Lemma 2.18, $R(\hat{A} - \lambda\hat{B}) = \phi(R(A - \lambda B))$ so $\phi(R(A - \lambda B))$ is closed. Now $B(N(A)) \subset R(A - \lambda B)$ by Lemma 2.19 and therefore, by Lemma 2.13, $R(A - \lambda B)$ is closed.

To prove that $N(A - \lambda B) = N(A) \cap N(B)$ we note that $N(A) \cap N(B) \subset N(A - \lambda B)$ for all λ. Now if $z \in N(A - \lambda B)$ then $Az = \lambda Bz$ and $\phi(Az) = \lambda\phi(Bz)$. Thus $\hat{A}z' = \lambda\hat{B}z'$. But $\hat{A} - \lambda\hat{B}$ is one-to-one so $z' = 0$. This implies that $z \in N(A)$ so $Az = 0$. Therefore $\lambda Bz = 0$ and $z \in N(B)$ since $\lambda \neq 0$. Thus $z \in N(A) \cap N(B)$.

Finally since $N(B) \subset J(A - \lambda B{:}B)$ and $N(A - \lambda B) \subset N(B)$, $N(A - \lambda B) \subset J(A - \lambda B{:}B)$ and the proof of Theorem 2.9 is complete.

A question which arises naturally at this point is the possibility of replacing the condition: $|Ax + Bw| \geq k|Ax|$ for all $x \in D(A)$ and for all $w \in N(A)$ by the weaker condition: $R(A) \cap B(N(A)) = \{0\}$ in Theorem 2.9. The following example demonstrates that this is not possible. Moreover the example shows that our general hypothesis is not sufficient to guarantee the existence of a neighborhood of 0 such that $R(A - \lambda B)$ is closed for all λ in the neighborhood.

Consider a Banach space ℓ_p where $1 \leq p < \infty$. We denote the elements in the space by $x = \{x(k)\}$. We define the operator A by $A(\{x(k)\}) = y = \{y(k)\}$ where $y(k) = x(k)$ for k odd and $y(k) = kx(k - 1)$ for k even. Thus $D(A) = \{x : x \in \ell_p,\ Ax \in \ell_p\}$ and $R(A) = \{y : y \in \ell_p,\ y(k) = ky(k - 1)$ for k even$\}$. Clearly $N(A) = \{x : x \in \ell_p,\ x(k) = 0$ for k odd$\}$. It is easy to show that $N(A)$ is closed. We define $B = I$ (the identity operator).

We shall first show that $R(A)$ is closed. Let $y \in \overline{R(A)}$. Then there exists a sequence $\{y_n\} \subset R(A)$ such that $y_n \to y$. We note that $|y_n - y| \geq |y_n(k) - y(k)|$ for each k. Suppose $y \notin R(A)$. Then there exists some even number $k = a$ such that $y(a) \neq ay(a - 1)$. Let $\delta = \frac{1}{2a}|ay(a - 1) - y(a)|$. Since $y_n \to y$ there exists a number N such that for all $n \geq N$,

$$|y_n(k) - y(k)| \leq |y_n - y| < \delta.$$

So $|y_n(k) - y(k)| < \delta$ for every k and for all $n \geq N$. Thus for $n \geq N$, $|y_n(a) - y(a)| < \delta$ and $|y_n(a - 1) - y(a - 1)| < \delta$

but

$$|y_n(a) - y(a)| = |y_n(a) - y(a) + ay(a - 1) - ay(a - 1)|$$

$$= |ay(a - 1) - y(a) + ay_n(a - 1) - ay(a - 1)|$$

since $y_n(a) = ay_n(a - 1)$ and

$$|y_n(a) - y(a)| \geq |ay(a - 1) - y(a)| - a|y_n(a - 1) - y(a - 1)|$$

$$\geq 2a\delta - a\delta = a\delta > \delta$$

which is a contradiction. Thus $y(k) = ky(k - 1)$ for all even k and therefore $y \in R(A)$.

To prove that A is closed we consider the quotient space $\ell_p/N(A)$ and the operator A'. It is easy to see that $|Ax| = |A'x'| \geq |x'|$ for all $x' \in D(A')$. Now let $x_n \longrightarrow x$ and $Ax_n \longrightarrow y$. Since $R(A)$ is closed $y \in R(A)$ so there exists $x_0 \in D(A)$ such that $y = Ax_0$. Now

$$|x' - x_0'| \leq |x' - x_n'| + |x_n' - x_0'| \leq |x' - x_n'| + |Ax_n - Ax_0|$$

from which it is clear that $x' = x_0'$ which implies that $x - x_0 \in N(A)$. So $x \in D(A)$ and $y = Ax$. Thus A is closed.

Next we show that there does not exist a positive number k such that $|Ax + Iw| \geq k|Ax|$ for all $x \in D(A)$ and all $w \in N(A)$. From the definition of A it is clear that $N(A) \cap R(A) = \{0\}$. Consider the sequences $\{w_n\} \subset N(A)$ and $\{x_n\} \subset D(A)$ where

$$w_n(k) = \begin{cases} 1 \text{ if } k = 2n \\ 0 \text{ if } k \neq 2n \end{cases}$$

and

$$x_n(k) = \begin{cases} 1/2n \text{ if } k = 2n - 1 \\ 0 \quad\ \text{ if } k \neq 2n - 1 \end{cases}$$

$|Ax_n - Iw_n| = \frac{1}{2n}$ and $|Ax_n| > 1$ for all n.

Finally we will show that for every λ which satisfies $0 < |\lambda| < 1$, $R(A - \lambda I)$ is not closed. $(A - \lambda I)x = y$ where $y(k) = x(k) - \lambda x(k)$ for k odd and $y(k) = kx(k - 1) - \lambda x(k)$ for k even. It is easy to see that $A - \lambda I$ is one-to-one for $0 < |\lambda| < 1$. Now consider the sequence $\{x_n\}$ where

$$x_n(k) = \begin{cases} \lambda/2n & \text{for } k = 2n - 1 \\ 1 & \text{for } k = 2n \\ 0 & \text{otherwise} \end{cases}$$

$|(A - \lambda I)x_n| = \left|\frac{(1 - \lambda)\lambda}{2n}\right| \longrightarrow 0$ as $n \to \infty$ while $|x_n| > 1$. Thus $(A - \lambda I)^{-1}$ is not continuous and therefore $D[(A - \lambda I)^{-1}] = R(A - \lambda I)$ is not closed for $0 < |\lambda| < 1$.

III. STABILITY OF $K(A - \lambda B:B)$

<u>Definition 3.0</u>: We define the linear manifold $K(A - \lambda B:B)$ to be the intersection of the linear manifolds,

$$(A - \lambda B)\left[M_n(A - \lambda B:B)\right],$$

for all non-negative integers n.

We continue to use the same general hypothesis as that of Chapter II. In this chapter we shall consider the stability of $K(A - \lambda B:B)$ as a function of λ for λ in certain regions of the complex plane. We will show such stability under the same type of conditions which insure that $R(A - \lambda B)$ is closed for all λ in a neighborhood of 0.

In order to simplify the notation we make the following abbreviations:

1. $K = K(A:B)$
2. $K(\lambda) = K(A - \lambda B:B)$
3. $J = J(A:B)$
4. $J(\lambda) = J(A - \lambda B:B)$
5. $M_n = M_n(A:B)$
6. $M_n(\lambda) = M_n(A - \lambda B:B)$
7. $\Upsilon = \Upsilon(A)$
8. $\Upsilon(\lambda) = \Upsilon(A - \lambda B)$

<u>Theorem 3.1</u>: $K(\lambda) = B(J(\lambda))$.

Proof: Without loss of generality we can assume that $\lambda = 0$. Then

$$J = \bigcap_{n=0}^{\infty} M_n = \bigcap_{n=1}^{\infty} M_n = \bigcap_{n=1}^{\infty} B^{-1}(AM_{n-1}) = B^{-1}(\bigcap_{n=0}^{\infty} AM_n) = B^{-1}(K)$$

and therefore $B(J) = K$.

Theorem 3.2: Let $N(A - \lambda_0 B) \subset J(\lambda_0)$. Suppose there exists a sequence $\{\lambda_j\}$ such that $\lambda_j \to \lambda_0$ ($\lambda_j \neq \lambda_0$ for every j) and $y \in R(A - \lambda_j B)$ for all j. Then $y \in K(\lambda_0)$.

Proof: The theorem will be proved in three sections. Again we can assume that $\lambda_0 = 0$.

(1) We first show that $\gamma(\lambda_j) \geq \gamma/2 > 0$. By possibly omitting a finite number of λ_j's we can assume that $|\lambda_j| \leq \gamma/(6\sigma + 2\tau\gamma)$ since γ is positive by Lemma 2.7. From Theorem 2.8, $\gamma(\lambda) \geq \gamma - (3\sigma + \tau\gamma)|\lambda|$. Therefore

$$\gamma(\lambda_j) \geq \gamma - (3\sigma + \tau\gamma)|\lambda_j| \geq \gamma - \frac{\gamma}{2} = \frac{\gamma}{2} > 0.$$

(2) Next we show that if the equation $z = (A - \lambda_j B)x$ is solvable for all j then $z \in R(A)$.

For each j let p_j be a solution of $z = (A - \lambda_j B)x$. Then by Lemma 2.7, $|z| = |(A - \lambda_j B)p_j| \geq \gamma(\lambda_j)|p_j'| \geq \frac{\gamma}{2}|p_j'|$ for all j, where $p_j' = \{p_j + t : t \in N(A - \lambda_j B)\}$ and $|p_j'| = \inf\{|p_j + t| : t \in N(A - \lambda_j B)\}$. For each j there exists $r_j \in p_j'$ such that $|r_j| \leq 2|p_j'| \leq \frac{4}{\gamma}|z|$. $(A - \lambda_j B)r_j =$

$(A - \lambda_j B)p_j$ since $r_j - p_j \in N(A - \lambda_j B)$. Thus $\{r_j\}$ is a bounded set of solutions of the equation $z = (A - \lambda_j B)x$. Now

$$|z| = |(A - \lambda_j B)r_j| \geq |Ar_j| - |\lambda_j||Br_j|$$
$$\geq |Ar_j| - |\lambda_j|(\sigma|r_j| + \tau|Ar_j|)$$

so

$$|z| + |\lambda_j|\sigma|r_j| \geq |Ar_j|(1 - |\lambda_j|\tau).$$

Since

$$|\lambda_j| \leq \frac{\gamma}{6\sigma + 2\tau\gamma} \leq \frac{1}{2\tau}, \quad (1 - |\lambda_j|\tau) \geq \frac{1}{2}$$

and

$$|Ar_j| \leq 2(|z| + |\lambda_j|\sigma|r_j|).$$

Thus $\{Ar_j\}$ is also bounded. Now $z = (A - \lambda_j B)r_j$ for all j so that $Ar_j - z = \lambda_j Br_j$. Then $|Ar_j - z| = |\lambda_j||Br_j|$ so that $|Ar_j - z| \leq |\lambda_j|(\sigma|r_j| + \tau|Ar_j|)$ from which it is clear that $\lim_{j \to \infty} Ar_j = z$. Since $R(A)$ is closed $z \in R(A)$.

(3) We are now ready to complete the proof of the theorem. Since $y \in R(A - \lambda_j B)$ for all j, the equation $y = (A - \lambda_j B)x$ is solvable for each j. By Section (2), $y \in R(A)$ so there exists $x_0 \in D(A)$ such that $y = Ax_0$. Now let u_j be any sequence of solutions of the equation $y = (A - \lambda_j B)x$ and note that $(A - \lambda_j B)\left[\frac{1}{\lambda_j}(u_j - x_0)\right] = Bx_0$. This equality is true for all j and thus the equation $(A - \lambda_j B)x = Bx_0$ is solvable for all j. By Section (2), $Bx_0 \in R(A)$ so there exists

$x_1 \in D(A)$ such that $Bx_0 = Ax_1$.

Now assume the induction hypothesis that $x_0, x_1, \cdots, x_n$ have been defined such that $Bx_{k-1} = Ax_k$, $k = 1, 2, \cdots, n$ and that the equation $(A - \lambda_j B)x = Bx_{n-1}$ is solvable for all j. Let $\{v_j\}$ be any sequence of solutions of the equation $(A - \lambda_j B)x = Bx_{n-1}$. Then $(A - \lambda_j B)\left[\frac{1}{\lambda_j}(v_j - x_n)\right] = \frac{1}{\lambda_j} Bx_{n-1} - \frac{1}{\lambda_j} Ax_n + Bx_n = Bx_n$. Thus $(A - \lambda_j B)x = Bx_n$ is solvable for all j and by Section (2), $Bx_n \in R(A)$. So there exists $x_{n+1} \in D(A)$ such that $Ax_{n+1} = Bx_n$.

Thus we have shown the existence of a sequence $x_0, x_1, \cdots$, such that $Ax_n = Bx_{n-1}$ for all n. Now let n be an arbitrary positive integer. Then $Ax_n = Bx_{n-1}$ implies that $x_{n-1} \in \{B^{-1}(Ax_n)\} \subset M_1$. By applying $Ax_{n-k+1} = Bx_{n-k}$ successively as $k = 1, 2, \cdots, n$ it is clear that $x_{n-k} \in \{B^{-1}(Ax_{n-k+1})\} \subset M_k$. Thus $x_0 \in M_n$ and since n is arbitrary, $x_0 \in M_n$ for all n. Therefore $y = Ax_0 \in AM_n$ for all n. So $y \in K$ and the proof is complete.

<u>Theorem 3.3</u>: Let $N(A - \lambda_0 B) \subset J(\lambda_0)$. Then $y \in R(A - \lambda B)$ for all $y \in K(\lambda_0)$ and all λ which satisfy the inequality:

$$|\lambda - \lambda_0| < \frac{\gamma(\lambda_0)}{3\sigma + \tau\,\gamma(\lambda_0)} .$$

Proof: Assume $\lambda_0 = 0$. We shall prove the theorem in two sections.

(1) We first prove that if $Au \in K$ then there exists $v \in D(A)$ such that $Av = Bu$, $Av \in K$ and

$$|v| \leq \frac{2}{\gamma}|Av| \leq \frac{2}{\gamma}(\sigma|u| + \tau|Au|).$$

If $Au \in K$ then $Au \in AM_n$ for all n. Let n be an arbitrary positive integer. Then there exists $x \in M_n$ such that $Au = Ax$. Thus $u - x \in N(A)$ but since $N(A) \subset M_n$ it follows that $u \in M_n$. Since n is arbitrary, $u \in M_n$ for all n so $u \in J$. By Theorem 3.1, $Bu \in K$ so there exists $w \in D(A)$ such that $Aw = Bu$. Now by Lemma 2.7 we have

$$\gamma|w'| \leq |Aw| = |Bu| \leq \sigma|u| + \tau|Au|.$$

From the definition of the norm $X/N(A)$ it is clear that there exists $v \in w'$ such that $|v| \leq 2|w'|$. Since $v \in w'$, $Av = Aw$ so $|v| \leq \frac{2}{\gamma}|Av| \leq \frac{2}{\gamma}(\sigma|u| + \tau|Au|)$. Since $Bu \in K$ and $Av = Bu$, $Av \in K$.

(2) We now complete the proof of the theorem. Since $y \in K$, there exists some $x_0 \in D(A)$ such that $y = Ax_0 \in K$. By Section (1) there exists x_1 such that $Bx_0 = Ax_1 \in K$ and

$$|x_1| \leq \frac{2}{\gamma}|Ax_1| \leq \frac{2}{\gamma}(\sigma|x_0| + \tau|Ax_0|).$$

Suppose that $x_0, x_1, \cdots, x_n$ have been chosen so that $Ax_i = Bx_{i-1}$ where $Ax_i \in K$ and

$$|x_i| \leq \frac{2}{\gamma}|Ax_i| \leq \frac{2}{\gamma}\left(\frac{2\sigma + \tau\gamma}{\gamma}\right)^{i-1}(\sigma|x_0| + \tau|Ax_0|)$$

for $i = 1, 2, \cdots, n$. Since $Ax_n \in K$, by Section (1) there exists $x_{n+1} \in D(A)$ such that $Ax_{n+1} = Bx_n$, $Ax_{n+1} \in K$ and $|x_{n+1}| \leq \frac{2}{\gamma}|Ax_{n+1}| \leq \frac{2}{\gamma}(\sigma|x_n| + \tau|Ax_n|)$. Then since $|x_n| \leq \frac{2}{\gamma}|Ax_n|$, we have

$$\frac{2}{\gamma}(\sigma|x_n| + \tau|Ax_n|) \leq \frac{2}{\gamma}(\frac{2\sigma}{\gamma}|Ax_n| + \tau|Ax_n|) = \frac{2}{\gamma}(\frac{2\sigma + \tau\gamma}{\gamma})|Ax_n|$$

and

$$|x_{n+1}| \leq \frac{2}{\gamma}|Ax_{n+1}| \leq \frac{2}{\gamma}(\frac{2\sigma + \tau\gamma}{\gamma})^n(\sigma|x_o| + \tau|Ax_o|).$$

Thus by induction we have shown the existence of a sequence with the following properties: (1) $y = Ax_o$, (2) $Bx_{n-1} = Ax_n$ for all n, (3) $|Ax_n| \leq (\frac{2\sigma+\tau\gamma}{\gamma})^{n-1}(\sigma|x_o| + \tau|Ax_o|)$ for all n.

Now consider the sequence $\{z_n\}$ where $z_n = x_o + \lambda x_1 + \cdots + \lambda^n x_n \in D(A)$. We see that $(A - \lambda B)z_n = Ax_o - \lambda^{n+1}Bx_n = y - \lambda^{n+1}Ax_{n+1}$ and therefore

$$\begin{aligned}|(A - \lambda B)z_n - y| &= |-\lambda^{n+1}Ax_{n+1}| \leq |\lambda|^{n+1}|Ax_{n+1}| \\ &\leq (\frac{\gamma}{3\sigma + \tau\gamma})^{n+1}(\frac{2\sigma + \tau\gamma}{\gamma})^n(\sigma|x_o| + \tau|Ax_o|) \\ &= \frac{\gamma}{3\sigma + \tau\gamma}(\frac{2\sigma + \tau\gamma}{3\sigma + \tau\gamma})^n(\sigma|x_o| + \tau|Ax_o|).\end{aligned}$$

Since $\frac{2\sigma + \tau\gamma}{3\sigma + \tau\gamma} < 1$ for $\sigma \neq 0$ it is clear that $y = \lim_{n\to\infty} (A - \lambda B)z_n$ and therefore $y \in R(A - \lambda B)$ because $R(A - \lambda B)$ is closed. If $\sigma = 0$, let λ be an arbitrary fixed complex number such that

$0 < |\lambda| < 1/\tau$. Let $\delta = \frac{\gamma(1 - \tau|\lambda|)}{2|\lambda|}$. Then $|\lambda| =$

$\frac{\gamma}{2\delta + \tau\gamma} < \frac{\gamma}{\delta + \tau\gamma}$. Now

$$|(A - \lambda B)z_n - y| \leq \frac{\gamma}{\delta + \tau\gamma}\left(\frac{\tau\gamma}{\delta + \tau\gamma}\right)^n (\sigma|x_0| + \tau|Ax_0|)$$

and since $\frac{\tau\gamma}{\delta + \tau\gamma} < 1$, $(A - \lambda B)z_n \to y$. Thus $y \in R(A - \lambda B)$ since $R(A - \lambda B)$ is closed.

<u>Definition 3.4</u>: We define G to be the set of all λ such that:

1. $A - \lambda B$ is a closed linear operator.
2. $R(A - \lambda B)$ is closed.
3. $N(A - \lambda B) \subset J(\lambda)$.
4. There exist non-negative numbers σ and τ such that $|Bx| \leq \sigma|x| + \tau|(A - \lambda B)x|$ holds for all $x \in D(A)$.

In order to see that G is an open set we prove the following lemma.

<u>Lemma 3.5</u>: If $|Bx| \leq \sigma|x| + \tau|(A - \lambda_0 B)x|$ holds for all $x \in D(A)$ where σ and τ are non-negative numbers, then for all λ such that $|\lambda - \lambda_0|\tau < 1$, there exist non-negative numbers $\bar{\sigma}$ and $\bar{\tau}$ such that $|Bx| \leq \bar{\sigma}|x| + \bar{\tau}|(A - \lambda B)x|$ holds for all $x \in D(A)$.

Proof: We can assume that $\lambda_0 = 0$. Let $x \in D(A)$. Then

$$|Bx| \leq \sigma|x| + \tau|Ax| = \sigma|x| + \tau|(A - \lambda B)x + \lambda Bx|$$

$$\leq \sigma|x| + \tau|(A - \lambda B)x| + |\lambda|\tau|Bx|$$

so that

$$|Bx|(1 - |\lambda|\tau) \leq \sigma|x| + \tau|(A - \lambda B)x|.$$

Thus

$$|Bx| \leq \bar{\sigma}|x| + \bar{\tau}|(A - \lambda B)x|$$

where

$$\bar{\sigma} = \frac{\sigma}{1 - \tau|\lambda|} \quad \text{and} \quad \bar{\tau} = \frac{\tau}{1 - \tau|\lambda|}.$$

<u>Notation 3.6</u>: It is clear from Theorem 2.8 and Lemma 3.5 that G is an open set. We shall use Γ to denote any open connected subset of G.

<u>Theorem 3.7</u>: $K(\lambda)$ is independent of λ in Γ.

Proof: For each $y \in Y$, let Γ_y be the set of all $\lambda \in \Gamma$ for which $y \in K(\lambda)$. If $\lambda_0 \in \Gamma \cap \bar{\Gamma}_y$ then there exists a sequence $\{\lambda_j\} \subset \Gamma_y$ such that $\lambda_j \longrightarrow \lambda_0$ and $y \in K(\lambda_j)$ for all λ_j. Thus $y \in R(A - \lambda_j B)$ for all λ_j and by Theorem 3.2, $y \in K(\lambda_0)$ so $\lambda_0 \in \Gamma_y$ and hence Γ_y is closed in Γ.

Now let $\lambda_0 \in \Gamma_y$. By Theorem 3.3, there exists a neighborhood of λ_0 which is contained in Γ such that $y \in R(A - \lambda B)$ for all λ in the neighborhood. By Theorem 3.2, $\lambda \in \Gamma_y$ for each λ in the neighborhood. Thus Γ_y is open in Γ.

Since Γ is connected, Γ_y is either all of Γ or it is empty.

<u>Theorem 3.8</u>: For each $y \in Y$, either $y \in K(\lambda) \subset R(A - \lambda B)$ for all $\lambda \in \Gamma$, or there is no $\lambda \in \Gamma$ for which $y \in K(\lambda)$ and the set of all $\lambda \in \Gamma$ such that $y \in R(A - \lambda B)$ has no limit point in Γ.

Proof: If $y \in K(\lambda_0)$ for some $\lambda_0 \in \Gamma$, $y \in K(\lambda)$ for all $\lambda \in \Gamma$ by Theorem 3.7.

Now suppose that $\lambda_0 \in \Gamma$ is a limit point for the set of all $\lambda \in \Gamma$ such that $y \in R(A - \lambda B)$. By Theorem 3.2, $y \in K(\lambda_0)$ and the proof is complete.

<u>Theorem 3.9</u>: $\alpha(A - \lambda B)$ and $\beta(A - \lambda B)$ are constant in Γ.

Proof: Let λ_0 be an arbitrary element of Γ and let $S = \{\lambda : \lambda \in \Gamma,\ \alpha(A - \lambda B) = \alpha(A - \lambda_0 B) \text{ and } \beta(A - \lambda B) = \beta(A - \lambda_0 B)\}$.

Let $\lambda_1 \in S$. Then since $\lambda_1 \in \Gamma$, $N(A - \lambda_1 B) \subset J(\lambda_1)$ so by Theorem 2.8 there exists a neighborhood of λ_1 for which $\alpha(A - \lambda B)$ and $\beta(A - \lambda B)$ are constant for all λ in the neighborhood. Thus it is clear that S is open in Γ.

Now let $\lambda_2 \in \bar{S} \cap \Gamma$. By Theorem 2.8 there exists a neighborhood of λ_2 such that $\alpha(A - \lambda B)$ and $\beta(A - \lambda B)$ are constant for all λ in the neighborhood. But every neighborhood of λ_2 contains points of S so it is clear that $\lambda_2 \in S$. Therefore S is closed in Γ.

Since S is not empty and is both open and closed in Γ, S must be equal to Γ.

All of the above theorems depend upon the condition $N(A - \lambda_0 B) \subset J(\lambda_0)$. It is interesting to note a few special cases where this condition is rather easily seen to hold. For simplicity we consider $\lambda_0 = 0$.

1. $\alpha(A) = 0$. If $\alpha(A) = 0$, $N(A) = \{0\}$ and since J is a linear manifold $N(A) \subset J$.
2. $\beta(A) = 0$. If $\beta(A) = 0$, $R(A) = Y$ so that $M_1 = B^{-1}(AM_0) = B^{-1}(Y) = D(B)$. By induction it is clear that $M_n = D(B)$ for all n. Thus $J = D(B) \supset D(A) \supset N(A)$.
3. $N(A) \subset N(B)$. It is clear that $N(B) \subset J$ and therefore $N(A) \subset J$.
4. $R(B) \subset R(A)$. This case is similar to (2). Again it is easy to show that $D(B) = J$.
5. $D(A) \subset M_1$. Assume $D(A) \subset M_n$. Then $AM_n = R(A)$ and $M_{n+1} = B^{-1}(R(A)) = M_1$ so $D(A) \subset M_{n+1}$. Thus $D(A) \subset M_n$ for all n and so $N(A) \subset D(A) \subset J$.

We now turn to the case where $\alpha(A - \lambda_0 B)$ or $\beta(A - \lambda_0 B)$ is finite. Again we shall state a result of Kato (6) without proof.

<u>Theorem 3.10</u> (6, p. 315): Let $\alpha(A - \lambda_0 B)$ or $\beta(A - \lambda_0 B)$ be finite and suppose $N(A - \lambda_0 B) \not\subset J(\lambda_0)$. Then there exists a positive number ρ and a positive integer r such that, for $0 < |\lambda - \lambda_0| < \rho$, $A - \lambda B$ is a closed linear operator with closed range and $\alpha(A - \lambda B) = \alpha(A - \lambda_0 B) - r$, $\beta(A - \lambda B) = \beta(A - \lambda_0 B) - r$ and $N(A - \lambda B) \subset J(\lambda)$.

<u>Definition 3.11</u>: We define G_1 to be the set of all λ such

that:

1. $A - \lambda B$ is a closed linear operator.
2. $R(A - \lambda B)$ is closed.
3. $\alpha(A - \lambda B)$ or $\beta(A - \lambda B)$ is finite.
4. There exist non-negative numbers σ and τ such that $|Bx| \leq \sigma|x| + \tau|(A - \lambda B)x|$ holds for all $x \in D(A)$.

<u>Notation 3.12</u>: From Theorem 3.10 and Lemma 3.5 it is clear that G_1 is an open set. We use Γ_1 to denote any connected open subset of G_1.

<u>Notation 3.13</u>: For any Γ_1 we let Γ_0 denote the set of all $\lambda \in \Gamma_1$ for which $N(A - \lambda B) \not\subset J(\lambda)$.

<u>Lemma 3.14</u>: The set Γ_0 has no limit point in Γ_1.

Proof: Let $\lambda \in \Gamma_1$. If $\lambda \in \Gamma_0$ then by Theorem 3.10 there is a deleted neighborhood of λ which contains only points of $\Gamma_1 - \Gamma_0$. If $\lambda \in \Gamma_1 - \Gamma_0$ then $N(A - \lambda B) \subset J(\lambda)$ and by Theorem 2.8, there is a neighborhood of λ which contains no points of Γ_0 so Γ_0 has no limit point in Γ_1.

<u>Theorem 3.15</u>:[1] $K(\lambda)$ is independent of λ in $\Gamma_1 - \Gamma_0$. Furthermore $\alpha(A - \lambda B)$ and $\beta(A - \lambda B)$ are constant for all λ in $\Gamma_1 - \Gamma_0$.

Proof: $\Gamma_1 - \Gamma_0$ is an open connected subset of G (see

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[1] In the special case where $X = Y$ and $B = I$ (the identity operator) this theorem is known. See (2, p. 242).

Definition 3.4) so the theorem follows directly from Theorem 3.7 and Theorem 3.9.

<u>Theorem 3.16</u>: For each $y \in Y$, either $y \in K(\lambda)$ for all $\lambda \in \Gamma_1 - \Gamma_o$, or there is no $\lambda \in \Gamma_1 - \Gamma_o$ for which $y \in K(\lambda)$ and the set of all $\lambda \in \Gamma_1 - \Gamma_o$ such that $y \in R(A - \lambda B)$ has no limit point in $\Gamma_1 - \Gamma_o$.

Proof: This follows directly from Theorem 3.8.

We now return to the case where $B(N(A))$ is closed and $|Ax + Bw| \geq k|Ax|$ for all $x \in D(A)$ and for all $w \in N(A)$ which was considered in Theorem 2.10. In Lemma 3.17 and Lemma 3.18 the operator $\hat{A}$ is that defined by Definition 2.11 and $\hat{B}$ is the operator defined by Definition 2.12. We shall use the notation $\hat{M}_n$ for $M_n(\hat{A}:\hat{B})$ in order to distinguish these linear manifolds from $M_n(A:B)$. We note that $\hat{M}_0 = X/N(A)$.

<u>Lemma 3.17</u>: Let $B(N(A))$ be closed and suppose there exists $k > 0$ such that $|Ax + Bw| \geq k|Ax|$ for all $x \in D(A)$ and for all $w \in N(A)$. Then $\hat{A}\hat{M}_n = \emptyset(AM_n)$ for all n.

Proof: The equality is obvious for $n = 0$. Assume that $\hat{A}\hat{M}_m = \emptyset(AM_m)$ for a non-negative integer m and let $y' \in \hat{A}\hat{M}_{m+1}$. Then there exists $x_o' \in \hat{M}_{m+1}$ such that $y' = \hat{A}x_o'$.

$x_o' \in \hat{B}^{-1}(\hat{A}\hat{M}_m)$ which implies that $\hat{B}x_o' = \hat{A}x_1'$ for some $x_1' \in \hat{M}_m$. But $\hat{A}\hat{M}_m = \emptyset(AM_m)$ by the induction hypothesis so there exists $x_2 \in M_m$ such that $\hat{A}x_1' = \emptyset(Ax_2)$. Let $x_o \in x_o'$.

By definition of $\hat{B}$, $\hat{B}x_0' = \phi(Bx_0)$. Thus $\phi(Bx_0) = \phi(Ax_2)$ which implies that $Bx_0 - Ax_2 = Bw$ for some $w \in N(A)$. So $B(x_0 - w) = Ax_2$ and $x_0 - w \in \{B^{-1}(Ax_2)\} \subset M_{m+1}$. Now $A(x_0 - w) = Ax_0 \in AM_{m+1}$ which implies that $\phi(Ax_0) \in \phi(AM_{m+1})$. But $\phi(Ax_0) = \hat{A}x_0' = y'$ so $\hat{A}\hat{M}_{m+1} \subset \phi(AM_{m+1})$.

Now suppose $y' \in \phi(AM_{m+1})$. There exists $x_0 \in M_{m+1}$ such that $y' = \phi(Ax_0)$. $x_0 \in M_{m+1} = B^{-1}(AM_m)$ implies that $Bx_0 = Ax_1$ for some $x_1 \in M_m$. Again by the induction hypothesis $\phi(AM_m) = \hat{A}\hat{M}_m$ so there exists $x_2' \in \hat{M}_m$ such that $\phi(Ax_1) = \hat{A}x_2'$. Now $\phi(Bx_0) = \hat{B}x_0'$ where $x_0' = \theta(x_0)$ so $\hat{B}x_0' = \hat{A}x_2'$ which implies that $x_0' \in \{\hat{B}^{-1}(\hat{A}x_2')\} \subset \hat{M}_{m+1}$ and therefore $\hat{A}x_0' \in \hat{A}\hat{M}_{m+1}$. But $y' = \phi(Ax_0) = \hat{A}x_0'$ and consequently $\phi(AM_{m+1}) \subset \hat{A}\hat{M}_{m+1}$. Thus $\phi(AM_{m+1}) = \hat{A}\hat{M}_{m+1}$ and by induction $\phi(AM_n) = \hat{A}\hat{M}_n$ for all n.

<u>Lemma 3.18</u>: Let $B(N(A))$ be closed and suppose there exists $k > 0$ such that $|Ax + Bw| \geq k|Ax|$ for all $x \in D(A)$ and all $w \in N(A)$. Then $\phi(K(A:B)) = K(\hat{A}:\hat{B})$.

Proof: $\phi(K(A:B)) = \phi(\bigcap_{n=0}^{\infty} AM_n) \subset \bigcap_{n=0}^{\infty} \phi(AM_n)$. But $\phi(AM_n) = \hat{A}\hat{M}_n$ for all n by Lemma 3.17 so $\bigcap_{n=0}^{\infty} \phi(AM_n) = \bigcap_{n=0}^{\infty} \hat{A}\hat{M}_n = K(\hat{A}:\hat{B})$. Thus $\phi(K(A:B)) \subset K(\hat{A}:\hat{B})$.

Now let $y' \in K(\hat{A}:\hat{B}) = \bigcap_{n=0}^{\infty} \phi(AM_n)$. Then $y' \in \phi(AM_n)$ for all n. Let n be an arbitrary positive integer. Then $y' \in \phi(AM_n)$ implies the existence of $y_n \in AM_n$ such that $y' = \phi(y_n)$. Now $y' \in \phi(AM_0)$ so there exists $y_0 \in AM_0$ such that

$y' = \phi(y_o) = \phi(y_n)$. Thus $y_n - y_o \in B(N(A))$. But $y_n - y_o \in R(A)$ and since $B(N(A)) \cap R(A) = \{0\}$, $y_n = y_o$. Since n is arbitrary, $y_o \in AM_n$ for all n so that $y_o \in K(A:B)$. Thus $y' = \phi(y_o) \in \phi(K(A:B))$ so $K(\hat{A}:\hat{B}) \subset \phi(K(A:B))$.

Lemma 3.19: Let $B(N(A))$ be closed and let $|Ax + Bw| \geq k|Ax|$ for all $x \in D(A)$ and all $w \in N(A)$ where $k > 0$. Then $y \in R(A - \lambda B)$ for all $y \in K$ and all λ such that $|\lambda| < \frac{k\gamma}{3\sigma + \tau\gamma}$.

Proof: If $y \in K$ then $y' = \phi(y) \in K(\hat{A}:\hat{B})$ by Lemma 3.18. $\hat{A}$ and $\hat{B}$ satisfy the hypothesis of Theorem 3.3 so $y' \in R(\hat{A} - \lambda\hat{B})$ for all λ which satisfy

$$|\lambda| < \frac{\gamma(\hat{A})}{3\sigma + \frac{\tau}{k}\gamma(\hat{A})} .$$

From Lemma 2.15(c) it is clear that $\gamma(\hat{A}) \geq k\,\gamma(A)$ so that

$$\frac{k\gamma}{3\sigma + \tau\gamma} \leq \frac{\gamma(\hat{A})}{3\sigma + \frac{\tau}{k}\gamma(\hat{A})}$$

By Lemma 2.18 $R(\hat{A} - \lambda\hat{B}) = \phi(R(A - \lambda B))$ so there exists $y_o \in R(A - \lambda B)$ such that $y' = \phi(y_o)$. Thus $y - y_o \in B(N(A))$ but $B(N(A)) \subset R(A - \lambda B)$ by Lemma 2.19 and therefore $y \in R(A - \lambda B)$.

Theorem 3.20: Let $B(N(A))$ be closed and let $|Ax + Bw| \geq k|Ax|$ for all $x \in D(A)$ and all $w \in N(A)$ where $k > 0$. Then for all λ such that $0 < |\lambda| < \frac{k\gamma}{\sigma + \tau\gamma}$, $K(\lambda)$ is independent of λ. Furthermore $K \oplus B(N(A)) = K(\lambda)$.

Proof: From Theorem 2.10, $N(A - \lambda B) \subset J(\lambda)$ for $0 < |\lambda| < \frac{k\Upsilon}{\sigma + \tau\Upsilon}$. Therefore by Theorem 3.7 we see that $K(\lambda)$ is independent of λ in this neighborhood.

To prove that $K \oplus B(N(A)) = K(\lambda)$ let $y \in K$. By Lemma 3.19, $y \in R(A - \lambda B)$ for $|\lambda| < \frac{k\Upsilon}{3\sigma + \tau\Upsilon}$. Choose $\lambda_1 \neq 0$ which satisfies this inequality. By Theorem 3.2, $y \in K(\lambda_1)$. But $K(\lambda_1) = K(\lambda)$ for $0 < |\lambda| < \frac{k\Upsilon}{\sigma + \tau\Upsilon}$ since we have shown $K(\lambda)$ to be independent of λ. Thus $K \subset K(\lambda)$. Now by Lemma 2.19, $B(N(A)) \subset R(A - \lambda B)$ for all $\lambda \neq 0$ so by again using Theorem 3.2 it is clear that $B(N(A)) \subset K(\lambda)$. Since $R(A) \cap B(N(A)) = \{0\}$, we see that $K \cap B(N(A)) = \{0\}$. Thus $K \oplus B(N(A)) \subset K(\lambda)$ since $K(\lambda)$ is a linear manifold.

Now suppose $y \in K(\lambda)$. Then $y \in R(A - \lambda B)$ for $0 < |\lambda| < \frac{k\Upsilon}{\sigma + \tau\Upsilon}$. Thus $\phi(y) \in \phi(R(A - \lambda B)) = R(\hat{A} - \lambda\hat{B})$ and by Theorem 3.2, $\phi(y) \in K(\hat{A}:\hat{B})$. By Lemma 3.18, $\phi(y) \in \phi(K)$ which implies the existence of $y_0 \in K$ such that $\phi(y) = \phi(y_0)$. So $y - y_0 = Bw$ where $w \in N(A)$ and therefore $y = y_0 + Bw \in K \oplus B(N(A))$. Thus $K(\lambda) \subset K \oplus B(N(A))$ and the proof is complete.

IV. SOME CONDITIONS UNDER WHICH $K(A - \lambda B:B)$ IS CLOSED

We shall continue to use the same general hypothesis as that introduced in Chapter II. In Chapter III we discussed the stability of the linear manifold $K(\lambda)$ under various conditions. In this chapter we shall show that under these same conditions the linear manifolds AM_n are closed for all n and hence K is closed. As in Chapter II we are considering $\lambda_0 = 0$.

<u>Lemma 4.0</u>: If $AM_n + B(N(A))$ is a closed linear manifold for a non-negative integer n, then AM_{n+1} is closed.

Proof: Let $y \in \overline{AM}_{n+1}$. Then there exists a sequence $\{x_n\} \subset M_{n+1} \cap D(A)$ such that $Ax_n \to y$. By Lemma 2.7, $|Ax_n - Ax_m| = |A'x_n' - A'x_m'| \geq \gamma|x_n' - x_m'|$ where x_n' and x_m' represent the equivalence classes of $X/N(A)$ containing x_n and x_m respectively. From the above inequality it is clear that $\{x_n'\}$ is a Cauchy sequence. Let $x' = \lim_{n \to \infty} x_n'$ and let x be an arbitrary element of x'. From the definition of the norm in $X/N(A)$ there exists a sequence $\{w_n\} \subset N(A)$ such that $x_n + w_n \to x$. $A(x_n + w_n) = Ax_n \to y$ and since A is closed, $x \in D(A)$ and $y = Ax$. Since $\{x_n\} \subset M_{n+1} = B^{-1}(AM_n)$ it is clear that $\{Bx_n\} \subset AM_n$. Now

$$|B(x_n + w_n) - Bx| \leq \sigma|(x_n + w_n) - x| + \tau|A(x_n + w_n) - Ax|$$

and therefore $B(x_n + w_n) \to Bx$. But

$$\{B(x_n + w_n)\} \subset AM_n + B(N(A))$$

which is closed by hypothesis so $Bx \in AM_n + B(N(A))$. Thus there exist $\bar{x} \in M_n$ and $\bar{w} \in N(A)$ such that $Bx = A\bar{x} + B\bar{w}$. So $B(x - \bar{w}) = A\bar{x} \in AM_n$. Therefore $x - \bar{w} \in B^{-1}(AM_n) = M_{n+1}$ and $y = Ax = A(x - \bar{w}) \in AM_{n+1}$.

<u>Theorem 4.1</u>: If $N(A) \subset J$ then AM_n is closed for all n and K is closed.

Proof: $R(A) = AM_0$ is closed from our general hypothesis. For m a non-negative integer assume the induction hypothesis that AM_m is closed. Since $N(A) \subset J$, $N(A) \subset M_{m+1} = B^{-1}(AM_m)$ so $B(N(A)) \subset AM_m$ and thus $AM_m + B(N(A)) = AM_m$ is closed. By Lemma 4.0, AM_{m+1} is closed. Therefore AM_n is closed for all n. Clearly $K = \bigcap_{n=0}^{\infty} AM_n$ is closed and the proof is complete.

<u>Theorem 4.2</u>: Suppose $B(N(A))$ is closed and there exists $k > 0$ such that $|Ax + Bw| \geq k|Ax|$ for all $x \in D(A)$ and all $w \in N(A)$. Then AM_n is closed for all n and hence K is closed.

Proof: AM_0 is closed by our general hypothesis. Assume that AM_m is closed for a non-negative integer m. $AM_m \subset R(A)$ so that $|Ax + Bw| \geq k|Ax|$ for all $x \in M_m$ and all $w \in N(A)$. By Lemma 2.10, $AM_m + B(N(A))$ is closed. Thus by Lemma 4.0, AM_{m+1} is closed. So by induction AM_n is closed for all n.

<u>Theorem 4.3</u>: If $\alpha(A)$ is finite then AM_n is closed for all n and K is closed.

Proof: Again AM_0 is closed by hypothesis. If m is a non-negative integer such that AM_m is closed then $AM_m + B(N(A))$ is closed since it is at most a finite dimensional extension of AM_m. Applying Lemma 4.0, AM_{m+1} is closed. Thus AM_n is closed for all n and K is closed.

Before considering the case where $\beta(A)$ is finite we need some preliminary lemmas. Propositions similar to Lemma 4.5 and Lemma 4.6 are stated without proof by Kato (6, p. 271). We shall use the abbreviation dim S to represent the dimension of the linear space S.

The following lemma is obvious.

<u>Lemma 4.4</u>: Let S be a linear space and L any linear manifold contained in S. Let y denote any element of S and let y' denote the equivalence class in the quotient space S/L to which y belongs. A set $\{y_j : j = 1,2,\cdots,n\}$ is linearly independent modulo L if and only if $\{y'_j : j = 1,2,\cdots,n\}$ is a linearly independent set in the quotient space S/L.

<u>Lemma 4.5</u>: Let C be the linear operator defined by restricting B to D(A) and let L be any linear manifold in Y such that dim Y/L is finite. Then

$$\dim \frac{D(A)}{C^{-1}(L)} \leq \dim \frac{Y}{L}$$

Proof: Let $\{x'_j : j = 1,2,\cdots,n\}$ be any linearly independent set in $\frac{D(A)}{C^{-1}(L)}$ and let $x_j \in x_j'$, $j = 1, 2, \cdots, n$. By Lemma 4.4, $\{x_j : j = 1,2,\cdots,n\}$ is linearly independent modulo $C^{-1}(L)$. Let $y_j = Cx_j$, $j = 1, 2, \cdots, n$ and let y_j' be the equivalence class in Y/L containing y_j for each j. Consider scalars α_j, $j = 1, 2, \cdots, n$ such that $\sum_{j=1}^{n} \alpha_j y_j' = 0$. This implies that $C \sum_{j=1}^{n} \alpha_j x_j = \sum_{j=1}^{n} \alpha_j C x_j = \sum_{j=1}^{n} \alpha_j y_j \in L$ so that $\sum_{j=1}^{n} \alpha_j x_j \in C^{-1}(L)$. Therefore $\alpha_j = 0$, $j = 1, 2, \cdots, n$ so that $\{y_j' : j = 1,2,\cdots,n\}$ is a linearly independent set in Y/L from which it is clear that $n \leq \dim Y/L$ and the conclusion follows immediately.

<u>Lemma 4.6</u>: Let $\beta(A)$ be finite and let $L \subset D(A)$ be a linear manifold such that $\dim D(A)/L$ is finite. Then

$$\dim \frac{Y}{AL} \leq \dim \frac{D(A)}{L} + \beta(A).$$

Proof: Since $R(A) \subset Y$, $R(A)/AL$ is a subspace of Y/AL. Let $S = \{y_j' : j = 1,2,\cdots,n\}$ be any linearly independent set in $R(A)/AL$. Let $y_j \in y_j'$, $j = 1, 2, \cdots, n$. Now $y_j \in R(A)$ so $y_j = Ax_j$, $j = 1, 2, \cdots, n$. Let x_j' be the equivalence class in $D(A)/L$ to which x_j belongs. Consider scalars α_j, $j = 1, 2, \cdots, n$ such that $\sum_{j=1}^{n} \alpha_j x_j' = 0$. Then $\sum_{j=1}^{n} \alpha_j x_j \in L$ which implies that $A(\sum_{j=1}^{n} \alpha_j x_j) = \sum_{j=1}^{n} \alpha_j A x_j \in AL$ so that $\sum_{j=1}^{n} \alpha_j y_j' = 0$.

Thus we see that $\alpha_j = 0$, $j = 1, 2, \cdots, n$ and the set of x_j', $j = 1, 2, \cdots, n$ is a linearly independent set in $D(A)/L$. Therefore it is clear that $\dim \frac{R(A)}{AL} \leq \dim \frac{D(A)}{L}$ and $\dim \frac{D(A)}{L}$ is finite by hypothesis.

We shall now assume that $\dim \frac{R(A)}{AL} = n$ and we can also assume that the set S is a basis for $R(A)/AL$. Let S be augmented by m vectors which are linearly independent modulo AL so that the set $\{y_j':j = 1,2,\cdots,n+m\}$ is a linearly independent set in Y/AL. Consider the set $\{y_{n+i}':i = 1,2,\cdots,m\}$ and let $y_{n+i} \in y_{n+i}'$. Let δ_i, $i = 1, 2, \cdots, m$ be any set of scalars such that $y = \sum_{i=1}^{m} \delta_i y_{n+i} \in R(A)$. Then $y' \in R(A)/AL$ and so there exist scalars λ_j, $j = 1, 2, \cdots, n$ such that $y' = \sum_{i=1}^{m} \delta_i y_{n+i}' = \sum_{j=1}^{n} \lambda_j y_j'$ since the set S spans $R(A)/AL$. But $\{y_j':j = 1,2,\cdots,n+m\}$ is a linearly independent set in Y/AL so $\delta_i = 0$, $i = 1, 2, \cdots, m$. Therefore $\{y_{n+i}:i = 1,2,\cdots,m\}$ is linearly independent modulo $R(A)$ by Lemma 4.4 and

$$m \leq \dim \frac{Y}{R(A)} = \beta(A).$$

It is now clear that $\dim \frac{Y}{AL} \leq \dim \frac{D(A)}{L} + \beta(A)$.

<u>Lemma 4.7</u>: If $\beta(A)$ is finite then $\dim \frac{Y}{AM_n} \leq (n + 1)\beta(A)$ holds for all n.

Proof: $\dim \frac{Y}{AM_0} = \beta(A)$ so the lemma is true for $n = 0$. For m a non-negative integer assume that

$$\dim \frac{Y}{AM_m} \leq (m + 1)\beta(A).$$

By Lemma 4.5,

$$\dim \frac{D(A)}{C^{-1}(AM_m)} \leq \dim \frac{Y}{AM_m}$$

so that

$$\dim \frac{D(A)}{C^{-1}(AM_m)} \leq (m + 1)\beta(A).$$

It is clear that $A[C^{-1}(AM_m)] = A[B^{-1}(AM_m)] = AM_{m+1}$. Thus by Lemma 4.6

$$\dim \frac{Y}{AM_{m+1}} \leq \dim \frac{D(A)}{C^{-1}(AM_m)} + \beta(A) \leq (m + 1)\beta(A) + \beta(A)$$

$$= (m + 2)\beta(A).$$

Therefore by induction the inequality holds for all non-negative integers.

<u>Theorem 4.8</u>: If $\beta(A)$ is finite then AM_n is closed for all n and K is closed.

Proof: AM_0 is closed by our general hypothesis. For m a non-negative integer assume that AM_m is closed. By Lemma 4.7 $\dim \frac{Y}{AM_m} \leq (m + 1)\beta(A)$ from which it is clear that $AM_m + B(N(A))$ is at most a finite dimensional extension of AM_m. Thus $AM_m + B(N(A))$ is closed and by Lemma 4.0, AM_{m+1} is closed.

Therefore we have shown by induction that AM_n is closed for all n and consequently K is closed.

V. A THEOREM ON THE EXISTENCE OF $(A - \lambda B)^{-1}$

In this chapter we give a sufficient condition for the existence of $(A - \lambda B)^{-1}$ for λ in a deleted neighborhood of 0. We believe this to be a new result. We again assume the general hypothesis with $\lambda_0 = 0$.

Theorem 5.0: Suppose there exists $k > 0$ such that $|w + x| \geq k|x|$ for all $w \in N(A)$ and for all $x \in J \cap D(A)$. Then there exists $\rho > 0$ such that $A - \lambda B$ is one-to-one for all λ which satisfy the inequality $0 < |\lambda| < \rho$.

Proof: Assume the theorem is false. Let $\{N_j\}$ be a sequence of neighborhoods such that $N_j = \{\lambda : 0 < |\lambda| < 1/j\}$. For each j there exists $x_j \neq 0$ and $\lambda_j \in N_j$ such that $Ax_j = \lambda_j Bx_j$. Clearly $|\lambda_j| \to 0$ as $j \to \infty$. We can assume that $\{x_j\}$ is a normalized sequence.

Now for j arbitrary, $x_j \in X = M_0$. For m a non-negative integer assume that $x_j \in M_m$. Then $x_j/\lambda_j \in M_m$ and $Bx_j = A\, x_j/\lambda_j \in AM_m$. Thus $x_j \in B^{-1}(AM_m) = M_{m+1}$. Therefore by induction $x_j \in M_n$ for all n which implies that $x_j \in J$. Since j is arbitrary $x_j \in J$ for all j.

Now

$$|Ax_j| = |\lambda_j Bx_j| \leq |\lambda_j|(\sigma|x_j| + \tau|Ax_j|)$$

so that

$$|Ax_j|(1 - |\lambda_j|\tau) \leq |\lambda_j|\sigma|x_j|.$$

From this inequality it is clear that $|Ax_j| \to 0$ as $j \to \infty$.

From Lemma 2.7,

$$\gamma|x_j'| \leq |A'x_j'| = |Ax_j| = |\lambda_j Bx_j| \leq |\lambda_j|(\sigma|x_j| + \tau|Ax_j|)$$

where x_j' is the equivalence class in $X/N(A)$ to which x_j belongs. Since $\gamma > 0$ we see by the inequality above that $|x_j'| \to 0$ as $j \to \infty$. Thus there exists a sequence $\{z_j\}$ such that $z_j \in x_j'$ for each j and $|z_j| \to 0$ as $j \to \infty$. Since $z_j \in x_j'$, there exists $w_j \in N(A)$ for each j such that $z_j - x_j = w_j$. Now since $x_j \in J \cap D(A)$, $|z_j| = |x_j + w_j| \geq k|x_j|$ by hypothesis. Thus $|z_j| \geq k$ which contradicts $|z_j| \to 0$ as $j \to \infty$ and the theorem is proved.

<u>Corollary 5.1</u>: If $J \cap D(A) = \{0\}$ then $A - \lambda B$ is one-to-one for all $\lambda \neq 0$.

Proof: In the proof of Theorem 5.0 we showed that $x_j \in J$ where $Ax_j = \lambda_j Bx_j$. It is clear by the same argument that if $x \in N(A - \lambda B)$ where $\lambda \neq 0$ then $x \in J$. Thus $N(A - \lambda B) \subset J$ for $\lambda \neq 0$. Since $J \cap D(A) = \{0\}$, $N(A - \lambda B) = \{0\}$ for every $\lambda \neq 0$.

VI. SOME ADDITIONAL RESULTS IN THE CASE WHERE $X = Y$

We shall now consider the concept of regular extension which is defined in the special case where $X = Y$ and B is the identity operator I. We continue to assume that $A - \lambda_0 I$ is a closed operator with closed range. After considering the main theorem of the chapter we present an example in the same setting which gives further information about the existence of a neighborhood of $\lambda_0 = 0$ such that $R(A - \lambda I)$ is closed for all λ in the neighborhood.

In the special case under consideration we see by Theorem 3.1 that $J(\lambda) = K(\lambda)$. We also note that the linear manifolds $M_n(\lambda)$ reduce to $R[(A - \lambda I)^n]$. Thus $K(A - \lambda I:I)$ now reduces to $\bigcap_{n=1}^{\infty} R[(A - \lambda I)^n]$. We find it convenient in this chapter to denote $K(A - \lambda I:I)$ by $K(A - \lambda I)$.

We shall now consider a pair of closed operators A_0 and A_1 in X such that $A_0 \subset A_1$.

<u>Definition 6.0</u>: A closed operator A in X is said to be an extension if $A_0 \subset A \subset A_1$.

<u>Definition 6.1</u>: An extension A is said to be regular at λ if $R(A - \lambda I) = R(A_1 - \lambda I)$ and $A - \lambda I$ has a bounded inverse.

<u>Definition 6.2</u>: An operator A is said to be a regular extension near λ_0 if for every λ in some neighborhood of λ_0, A is a regular extension at λ.

Homer (5, p. 416) has shown that if A is a regular extension near λ_0 then $N(A_1 - \lambda_0 I) \subset K(A_1 - \lambda_0 I)$. The following theorem is essentially the converse of the above proposition.

<u>Theorem 6.3</u>: If A is a regular extension at λ_0 and if $N(A_1 - \lambda_0 I) \subset K(A_1 - \lambda_0 I)$ then A is a regular extension near λ_0.

Proof: Assume that $\lambda_0 = 0$. Akhiezer and Glazman (1, p. 91) have shown that the set of all λ such that $A - \lambda I$ has a bounded inverse is an open set. Thus we need only show that $R(A - \lambda I) = R(A_1 - \lambda I)$ for all λ in some neighborhood of 0.

We shall first show that $D(A_1) = D(A) \oplus N(A_1)$. Let $x \in D(A_1)$. Then $A_1 x \in R(A_1) = R(A)$. So there exists $z \in D(A)$ such that $A_1 x = Az$. But since $A \subset A_1$, $x - z \in D(A_1)$ and we see that $A_1(x - z) = 0$. Thus $x - z \in N(A_1)$ from which it is clear that $D(A_1) \subset D(A) + N(A_1)$. Now suppose $D(A) \cap N(A_1) \neq \{0\}$ and let $0 \neq x \in D(A) \cap N(A_1)$. Then $Ax = A_1 x = 0$ which contradicts the fact that A has an inverse. It is clear that $D(A) \oplus N(A_1) \subset D(A_1)$ and thus $D(A) \oplus N(A_1) = D(A_1)$.

We next show that $K(A) = K(A_1)$. To do this we need only show that $R(A^n) = R(A_1^n)$. This is true by hypothesis for $n = 1$. For m a positive integer assume the induction hypothesis that $R(A^m) = R(A_1^m)$. Let $y \in R(A_1^{m+1})$. Then there exists some $x \in D(A_1)$ such that $y = A_1^{m+1} x = A_1(A_1^m x)$. Thus

$A_1^m x \in D(A_1)$ so that $A_1^m x = z + w$ for some $z \in D(A)$ and $w \in N(A_1)$. But $N(A_1) \subset R(A_1^m)$ since $N(A_1) \subset K(A_1) = \bigcap_{n=1}^{\infty} R(A_1^n)$. So $z = A_1^m x - w \in R(A_1^m)$. Therefore by the induction hypothesis $z \in R(A^m)$ so that there exists $\bar{z} \in D(A)$ such that $z = A^m \bar{z}$. Now $y = A_1(A_1^m x) = A_1(z + w) = Az = A^{m+1} \bar{z}$. Thus $y \in R(A^{m+1})$. It is clear that $R(A^{m+1}) \subset R(A_1^{m+1})$. Therefore $R(A^n) = R(A_1^n)$ for all n and $K(A) = K(A_1)$.

Finally we show that $R(A - \lambda I) = R(A_1 - \lambda I)$ for all λ in some neighborhood of 0. It is clear that $R(A - \lambda I) \subset R(A_1 - \lambda I)$ for all λ. Since $N(A_1) \subset K(A_1)$ by hypothesis, $N(A_1) \subset K(A)$. Since $N(A) = \{0\}$ we can apply Theorem 3.3 to the operator A and we have that $N(A_1) \subset R(A - \lambda I)$ for all λ in some neighborhood of 0. Let λ be an arbitrary but fixed complex number in this neighborhood and let $y \in R(A_1 - \lambda I)$. Then $y = (A_1 - \lambda I)x$ for some $x \in D(A_1)$ and $x = z + w$ for some $z \in D(A)$ and some $w \in N(A_1)$. Since $N(A_1) \subset R(A - \lambda I)$, $w = (A - \lambda I)\bar{z}$ for some $\bar{z} \in D(A)$. Now consider $z - \lambda\bar{z} \in D(A)$. Then $(A - \lambda I)(z - \lambda\bar{z}) = (A - \lambda I)z - \lambda w = Az - \lambda(z + w) = A_1 x - \lambda x = y$ so $y \in R(A - \lambda I)$ and the proof is complete.

Since the existence of a regular extension near λ is sufficient to guarantee that $N(A_1 - \lambda I) \subset K(A_1 - \lambda I)$ it is clear that Theorem 3.8 is a generalization of Homer's result on the stability of $K(A_1 - \lambda I)$ (5, p. 415).

We now present an example with the property that A is a closed operator with closed range, $N(A) \subset R(A)$ and $N(A) \cap K(A) = \{0\}$ but $R(A - \lambda I)$ is not closed for all non-zero λ.

Consider an ℓ_p space where $1 \leq p \leq \infty$. Again we denote an element in ℓ_p by $x = \{x(k)\}$. We define A as follows: $A(\{x_k\}) = y = \{y(k)\}$ where $y(k) = 0$ for k odd and $y(k) = (k - 1)[x(k - 1)]$ for k even. $D(A) = \{x : Ax \in \ell_p\}$. It is clear that $R(A) = \{y : y \in \ell_p$ and $y(k) = 0$ for k odd$\}$ and that $N(A) = \{w : w \in \ell_p$ and $w(k) = 0$ for k odd$\}$. Thus $N(A) = R(A)$. Consequently $R(A^2) = \{0\}$ so that $K(A) = \{0\}$. By Corollary 3.10 we see that $A - \lambda I$ will be one-to-one for all $\lambda \neq 0$. We will omit the straightforward proof that $R(A)$ and $N(A)$ are closed.

To show that A is closed we consider the quotient space $\ell_p/N(A)$ and the operator A'. It is easy to see that $|Ax| = |A'x'| \geq |x'|$ for all $x' \in D(A')$. Suppose $x_n \longrightarrow x$ and $Ax_n \longrightarrow y$. Then $y \in R(A)$ since $R(A)$ is closed so there exists $x_0 \in D(A)$ such that $y = Ax_0$. Now

$$|x' - x_0'| \leq |x' - x_n'| + |x_n' - x_0'| \leq |x' - x_n'| + |Ax_n - Ax_0|$$

from which it is clear that $x' = x_0'$ so that $x - x_0 \in N(A)$. Thus $x \in D(A)$ and $y = Ax$.

For any $\lambda \neq 0$ consider the sequence $\{x_n\}$ where

$$x_n(k) = \begin{cases} \dfrac{\lambda}{2n - 1} & \text{if } k = 2n - 1 \\ 1 & \text{if } k = 2n \\ 0 & \text{otherwise} \end{cases} \quad . \quad \text{Clearly } |x_n| \geq 1.$$

Now $|(A - \lambda I)x_n| = \left|\dfrac{\lambda^2}{2n - 1}\right| \longrightarrow 0$ as $n \longrightarrow \infty$. Thus $(A - \lambda I)^{-1}$ exists but is not bounded for any $\lambda \neq 0$ so $R(A - \lambda I)$ is not closed for any $\lambda \neq 0$.

We recall from Theorem 2.8 and Theorem 2.9 that $R(A - \lambda I)$ is closed for all λ in some neighborhood of 0 if $N(A) \subset K(A)$ or if for some $k > 0$ the condition $|y + w| \geq k|y|$ holds for all $y \in R(A)$ and all $w \in N(A)$. The previous example and the example of Chapter II give clear limitations of possible extensions of the results contained in Theorem 2.8 and Theorem 2.9.

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